NONSTEADY DIFFUSION WITH VARIABLE COEFFICIENTS IN THE BOUNDARY CONDITIONS

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A method of generalized integral transformations is used to solve the problem of nonsteady diffusion with time-dependent coefficients in the boundary conditions. Such an approach does not require a solution of an integral equation for the surface potential or of a time-dependent eigenvalue problem. A formal solution is obtained on the basis of an infinite system of ordinary differential equations. An example is considered and numerical results are discussed.

Generalized solutions for an extensive class of linear problems of diffusion, obtained using the method of integral transformations, have been given in [1]. Because of the limited applicability of that method, however, some problems of heat and mass transfer of practical importance were not considered. They include the diffusion problem with time-dependent coefficients in the boundary conditions.

In [2] such a class of problems has been formally reduced to an infinite system of ordinary differential equations. The formal solution obtain has been expanded to multilayered regions [3] and to variable coefficients [4]. The approach developed in [2] was then used to obtain numerical results for the problem of forced internal convection with variable heat-transfer coefficients in [5, 6], in which the corresponding time-dependent eigenvalue problem was solved and numerical integration was used for the more general case.

It has been suggested that the ideas of generalized integral transformations be used to solve diffusion problems with variable coefficients [7-9]. Such an approach would get around the problems associated with the time-dependence of the eigenvalue problem and the need to solve an integral equation for the surface potential. The problem is solved by reducing the appropriate system of ordinary differential equations. An approximate but explicit analytical solution is obtained on the basis of the domination of diagonal terms in the matrix of coefficients of that system.

In the analysis presented here, the approach developed in [7] is altered so that it can be extended to time-dependence coefficients. The numerical results obtained for the problem of the thermal conductivity of a plate for a variable Biot number are compared with those published in [10-12].

The starting point of our analysis will be the fairly general formulation

$$w(\mathbf{x})\frac{\partial T(\mathbf{x},\ t)}{\partial t} = \nabla K(\mathbf{x})\nabla T(\mathbf{x},\ t) - d(\mathbf{x})\,T(\mathbf{x},\ t) + P(\mathbf{x},\ t), \quad \mathbf{x}\in V,\ t>0, \tag{1a}$$

in a homogeneous finite region V with time-dependence coefficients $\alpha(\mathbf{x},\ t)$ or $\beta(\mathbf{x},\ t)$ under the boundary conditions

$$\alpha(\mathbf{x},t)T(\mathbf{x},t)+\beta(\mathbf{x})K(\mathbf{x})\frac{\partial T(\mathbf{x},t)}{\partial \mathbf{n}} = \Phi(\mathbf{x},t),\ \mathbf{x}\in S,\ t>0, \tag{1b}$$

and the initial condition

$$T(\mathbf{x},t) = f(\mathbf{x}),\ \mathbf{x}\in V,\ t=0, \tag{1c}$$

where $\partial/\partial\mathbf{n}$ denotes a derivative with respect to the outward normal to the boundary surface S.

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TABLE 1. Comparison of Different Low-Order Solutions and a Previously Published Approximate Solution for $Bi(t) = 1.2 - e^{-t}$

t	$\Phi(1, t)+1$			$\Phi(0, t)+1$		
	data of [10]	from Eq. (10a)	from Eq. (12b)	data of [10]	from Eq. (10a)	from Eq. (12b)
0,5	0,5560	0,5064* 0,5204** 0,5370***	0,5351 0,5350 0,5353	0,3971	0,4323 0,4026 0,3987	0,4044 0,4047 0,4045
1,0	0,7143	0,6449 0,6956 0,6910	0,6773 0,6777 0,6776	0,5691	0,5888 0,5596 0,5512	0,5408 0,5411 0,5409
1,5	0,8135	0,7638 0,7983 0,7906	0,7814 0,7817 0,7816	0,7016	0,7258 0,6900 0,6823	0,6701 0,6702 0,6700
2,0	0,8983	0,8503 0,8659 0,8594	0,8540 0,8541 0,8540	0,8151	0,8261 0,7877 0,7821	0,7718 0,7718 0,7717
2,5	0,9275	0,9079 0,9112 0,9065	0,9032 0,9032 0,9032	0,8779	0,8929 0,8571 0,8534	0,8456 0,8455 0,8454
3,0	0,9543	0,9444 0,9413 0,9382	0,9361 0,9361 0,9360	0,9281	0,9353 0,9047 0,9025	0,8968 0,8967 0,8967
3,5	0,9746	0,9668 0,9613 0,9593	0,9579 0,9579 0,9579	0,9555	0,9613 0,9369 0,9356	0,9316 0,9314 0,9314
4,0	0,9902	0,9803 0,9746 0,9733	0,9723 0,9723 0,9723	0,9822	0,9771 0,9583 0,9576	0,9548 0,9547 0,9547

Note. (+): complete solution with 80 terms (N = 80).
*$Bi_0^* = 0.2$.
**$Bi^* = 1.2$.
***$Bi^* = \overline{Bi}^*$.

It is impossible to solve the problem (1) by the classical method of integral transformation [1] because of the coefficient $\alpha(\mathbf{x}, t)$. Let us consider the spectral problem

$$\nabla K(\mathbf{x}) \nabla \psi(\mu_i, \mathbf{x}) + [\mu_i^2 w(\mathbf{x}) - d(\mathbf{x})] \psi(\mu_i, \mathbf{x}) = 0, \quad \mathbf{x} \in V, \tag{2a}$$

with the boundary conditions

$$\alpha^*(\mathbf{x}) \psi(\mu_i \mathbf{x}) + \beta(\mathbf{x}) K(\mathbf{x}) \frac{\partial \psi(\mu_i, \mathbf{x})}{\partial \mathbf{n}} = 0, \quad \mathbf{x} \in S, \tag{2b}$$

where $\alpha^*(\mathbf{x})$ is some constant that characterizes $\alpha(\mathbf{x}, t)$. It may be the coefficient averaged over time, its value at the initial or final time, etc.

The formulation (2) enables us to use the integral transformation

$$\overline{T}_i(t) = \int_V w(\mathbf{x}) k_i(\mathbf{x}) T(\mathbf{x}, t) dv \tag{3a}$$

with the inversion equation

$$T(\mathbf{x}, t) = \sum_{i=0}^{\infty} k_i(\mathbf{x}) \overline{T}_i(t), \tag{3b}$$

where the kernel of the transformation is defined by the equation

$$k_i(\mathbf{x}) = \frac{\psi(\mu_i, \mathbf{x})}{N_i^{1/2}}, \tag{3c}$$

TABLE 2. Convergence of the Complete Solution at the Plate Surface [$\Phi(1, t) + 1$]

t	N = 20	40	60	80
		$Bi^*=0{,}2$		
0,500	0,5343	0,5348	0,5350	0,5351
1,00	0,6766	0,6771	0,6772	0,6773
1,50	0,7811	0,7813	0,7814	0,7814
2,00	0,8539	0,8539	0,8539	0,8540
2,50	0,9032	0,9032	0,9032	0,9032
3,00	0,9362	0,9361	0,9361	0,9361
3,50	0,9581	0,9580	0,9579	0,9579
4,00	0,9725	0,9724	0,9723	0,9723
		$Bi^*=1{,}2$		
0,500	0,5342	0,5347	0,5349	0,5350
1,00	0,6782	0,6778	0,6777	0,6777
1,50	0,7822	0,7819	0,7818	0,7817
2,00	0,8545	0,8542	0,8541	0,8541
2,50	0,9035	0,9033	0,9032	0,9032
3,00	0,9362	0,9361	0,9361	0,9361
3,50	0,9580	0,9579	0,9579	0,9579
4,00	0,9724	0,9723	0,9723	0,9723
		$Bi^*=\overline{Bi}^*$		
0,500	0,5351	0,5352	0,5352	0,5353
1,00	0,6779	0,6777	0,6776	0,6776
1,50	0,7819	0,7817	0,7817	0,7816
2,00	0,8542	0,8541	0,8541	0,8540
2,50	0,9032	0,9032	0,9032	0,9032
3,00	0,9361	0,9361	0,9360	0,9360
3,50	0,9579	0,9579	0,9579	0,9579
4,00	0,9723	0,9723	0,9723	0,9723

and the normalizing integral is defined as

$$N_i = \int_V w(\mathbf{x})\,\psi^2(\mu_i, \mathbf{x})\,dv. \tag{3d}$$

After multiplication by $\infty_V K_i(\mathbf{x})dv$ and integration over the volume V, the initial problem (1) is transformed to

$$\frac{d\overline{T}_i(t)}{dt} + \mu_i^2\overline{T}_i(t) - \frac{1}{N_i^{1/2}}\int_S K(\mathbf{x})\left[\psi_i(\mathbf{x})\frac{\partial T(\mathbf{x},\ t)}{\partial \mathbf{n}} - T(\mathbf{x},\ t)\frac{\partial \psi_i(\mathbf{x})}{\partial \mathbf{n}}\right]ds = \overline{h}_i(t),\ \ t>0, \tag{4a}$$

where

$$\overline{h_i}(t) = \int_V k_i(\mathbf{x})\,P(\mathbf{x},\ t)\,dv. \tag{4b}$$

The surface integral in Eq. (4a) is found by multiplying Eq. (1b) by $\psi_i(\mathbf{x}_s)$, multiplying Eq. (2b) by $T(\mathbf{x}_s, t)$, and subtracting the results. We finally obtain

$$K(\mathbf{x})\left[\psi_i(\mathbf{x})\frac{\partial T(\mathbf{x},\ t)}{\partial \mathbf{n}} - T(\mathbf{x},\ t)\frac{\partial \psi_i(\mathbf{x})}{\partial \mathbf{n}}\right] = \frac{1}{\beta(\mathbf{x})}\left[\Phi(\mathbf{x},\ t)\,\psi_i(\mathbf{x}) + (\alpha^*(\mathbf{x}) - \alpha(\mathbf{x},\ t))\,\psi_i(\mathbf{x})\,T(\mathbf{x},\ t)\right] \quad \text{for } \mathbf{x} = \mathbf{x}_s. \tag{5}$$

Equation (4a) thus reduces to

$$\frac{d\overline{T}_i(t)}{dt} + \mu_i^2\overline{T}_i(t) - \frac{1}{N_i^{1/2}}\int_S \frac{1}{\beta(\mathbf{x})}(\alpha^*(\mathbf{x}) - \alpha(\mathbf{x},\ t))\,\psi_i(\mathbf{x})\,T(\mathbf{x},\ t)\,ds = \overline{g}_i(t),\ \ t>0, \tag{6a}$$

TABLE 3. Convergence of the Complete Solution at the Center of the Plate [$\Phi(0, t) + 1$]

t	N = 20	40	60	80
		$Bi^*=0,2$		
0,500	0,4044	0,4044	0,4044	0,4044
1,00	0,5410	0,5409	0,5409	0,5408
1,50	0,6706	0,6703	0,6701	0,6701
2,00	0,7725	0,7721	0,7719	0,7718
2,50	0,8463	0,8458	0,8457	0,8456
3,00	0,8974	0,8970	0,8969	0,8968
3,50	0,9321	0,9317	0,9316	0,9316
4,00	0,9552	0,9550	0,9549	0,9548
		$Bi^*=1,2$		
0,500	0,4056	0,4050	0,4048	0,4047
1,00	0,5422	0,5415	0,5412	0,5411
1,50	0,6711	0,6705	0,6703	0,6702
2,00	0,7724	0,7720	0,7719	0,7718
2,50	0,8459	0,8457	0,8456	0,8455
3,00	0,8970	0,8968	0,8967	0,8967
3,50	0,9316	0,9315	0,9315	0,9314
4,00	0,9548	0,9547	0,9547	0,9547
		$Bi^*=\overline{Bi}^*$		
0,500	0,4050	0,4047	0,4046	0,4045
1,00	0,5414	0,5411	0,5410	0,5409
1,50	0,6704	0,6702	0,6701	0,6700
2,00	0,7719	0,7718	0,7717	0,7717
2,50	0,8456	0,8455	0,8454	0,8454
3,00	0,8968	0,8967	0,8967	0,8967
3,50	0,9315	0,9314	0,9314	0,9314
4,00	0,9547	0,9547	0,9547	0,9547

where

$$\bar{g}_i(t) = \bar{h}_i(t) + \frac{1}{N_i^{1/2}} \int_S \frac{\Phi(\mathbf{x},\ t)}{\beta(\mathbf{x})} \psi_i(\mathbf{x})\, ds. \tag{6b}$$

Using the inversion equation (3b), we obtain an infinite system of ordinary equations

$$\frac{d\bar{T}_i(t)}{dt} + \mu_i^2 \bar{T}_i(t) - \sum_{j=1}^{\infty} A_{ij}^*(t)\, \bar{T}_j(t) = \bar{g}_i(t), \tag{7a}$$

$$i = 1, 2, \ldots,\ t > 0,$$

where

$$A_{ij}^*(t) = \frac{1}{N_i^{1/2} N_j^{1/2}} \int_S \frac{1}{\beta(\mathbf{x})} (\alpha^*(\mathbf{x}) - \alpha(\mathbf{x},\ t))\, \psi_i(\mathbf{x})\, \psi_j(\mathbf{x})\, ds. \tag{7b}$$

Transformation of the initial conditions yields

$$\bar{T}_i(0) = \bar{f}_i = \frac{1}{N_i^{1/2}} \int_V w(\mathbf{x}) f(\mathbf{x})\, \psi_i(\mathbf{x})\, dv. \tag{7c}$$

It is convenient to write the systems (7) in the matrix form

$$\mathbf{y}' + \mathbf{A}(t)\mathbf{y}(t) = \mathbf{g}(t), \tag{8a}$$

$$\mathbf{y}(0) = \mathbf{f}, \tag{8b}$$

where $\mathbf{A}(t) = \{a_{ij}\}$ is a symmetric $N \times N$ matrix with elements

$$a_{ij} = \delta_{ij}\mu_i^2 - A_{ij}^*(t), \tag{8c}$$

and

$$\delta_{ij} = \begin{cases} 0, & i \neq j, \\ 1, & i = j, \end{cases} \tag{8d}$$

$$\mathbf{y} = \{\overline{T}_1(t),\ \overline{T}_2(t),\ \ldots,\ \overline{T}_N(t)\}^T, \tag{8e}$$

$$\mathbf{g} = \{\overline{g}_1(t),\ \overline{g}_2(t),\ \ldots,\ \overline{g}_N(t)\}^T, \tag{8f}$$

$$\mathbf{f} = \{\overline{f}_1, \overline{f}_2, \ldots, \overline{f}_N\}^T. \tag{8g}$$

Here N must be large enough to ensure satisfactory convergence, as noted in [7].

Using the DGEAR procedure in the IMSL software package [13], we may easily obtain the vector of the transformed potential system (8). Subsequent application of the inversion equation (3b) enables us to calculate the required potential $T(\mathbf{x}, t)$.

An approximate but explicit analytical solution to the system (8) has also been suggested in [7]. Using the dominant role of the main diagonal of the matrix in the summation in Eq. (7a), we can take $i = j$, which leads to the system

$$\frac{d\overline{T}_{i,l}}{dt} + (\mu_i^2 - A_{ii}^*(t))\,\overline{T}_{i,l} = \overline{g}_i(t),\quad i = 1,2,\ \ldots,\ t > 0, \tag{9a}$$

$$\overline{T}_{i,l}(0) = \overline{f}_i, \tag{9b}$$

which has the solution

$$\overline{T}_{i,l}(t) = \overline{f}_i \exp\left[-\int_0^t a_{ii}(t')\,dt'\right] + \int_0^t g_i(t') \exp\left[-\int_{t'}^t a_{ii}(t'')\,dt''\right] dt', \tag{10a}$$

where

$$A_{ii}^*(t) = \frac{1}{N_i}\int_S \frac{1}{\beta(\mathbf{x})}\,(\alpha^*(\mathbf{x}) - \alpha(\mathbf{x}, t))\,\psi_i^2(\mathbf{x})\,ds, \tag{10b}$$

$$a_{ii}(t) = \mu_i^2 - A_{ii}^*(t). \tag{10c}$$

Such an explicit solution, and convenient for practical purposes, lets us obtain results that are more or less precise in some time interval, depending on the values of the nondiagonal elements $A_{ij}^*(t)$ and the way in which $\alpha^*(\mathbf{x})$ is chosen. Here the influence of the nondiagonal terms on the solution can be taken into account approximately using one analytical iteration of the complete system. The resulting system has the form

$$\frac{d\overline{T}_{i,h}}{dt} + (\mu_i^2 - A_{ii}^*(t))\,\overline{T}_{i,h} = \overline{G}_i(t),\quad i = 1,2,\ \ldots,\ t > 0, \tag{11a}$$

$$\overline{T}_{i,h}(0) = \overline{f}_i, \tag{11b}$$

where

$$\overline{G}_i(t) = \overline{g}_i(t) + \sum_{\substack{j=1 \\ j \neq i}}^{\infty} A_{ij}^*(t)\,\overline{T}_{j,l}(t). \tag{11c}$$

Its solution can be written in the form

$$\overline{T}_{i,h}(t) = \overline{T}_{i,l}(t) + \overline{T}_{i,c}(t) \tag{12a}$$

with a correction term

$$\overline{T}_{i,c}(t) = \int_0^t \left(\sum_{\substack{j=1 \\ j \neq i}}^{\infty} A_{ij}^*(t') \, \overline{T}_{j,l}(t') \right) \exp\left[-\int_{t'}^{t} a_{ii}(t'')\, dt'' \right] dt'. \tag{12b}$$

We may assume that such an approach increases the accuracy of the results and permits the use of explicit solutions for a wider time interval and a larger range of boundary coefficients.

It should be noted that for $\alpha = \alpha(\mathbf{x})$ not dependent on time, each of these solutions leads to the exact results obtained in [1].

To illustrate the approach suggested here, let us consider the heating of a flat wall with a boundary conditio of the third kind and a variable Biot number [10, 11]. We formulate the problem in the dimensionless form

$$\frac{d\Phi(x, t)}{\partial t} = \frac{\partial^2 \Phi(x, t)}{\partial x^2}, \quad 0 < x < 1, \ t > 0, \tag{13a}$$

with the boundary conditions

$$\frac{\partial \Phi(x, t)}{\partial x} = 0, \ x = 0, \ t > 0, \tag{13b}$$

$$\frac{\partial \Phi(x, t)}{\partial x} + \mathrm{Bi}(t)\, \Phi(x, t) = 0, \ x = 1, \ t > 0, \tag{13c}$$

and the initial conditions

$$\Phi(x, 0) = \Phi_0, \ 0 \leqslant x \leqslant 1. \tag{13d}$$

The solution of the corresponding eigenvalue problem is

$$\psi(\mu_i, x) = \cos(\mu_i x), \tag{14a}$$

$$N_i = \frac{1}{2}\left[1 + \frac{\sin(2\mu_i)}{2\mu_i}\right], \tag{14b}$$

while μ_i are roots of the characteristic equation

$$\mu_i \mathrm{tg}\, \mu_i = \mathrm{Bi}^*. \tag{14c}$$

A comparison with the system (7) enables to write

$$\overline{g}_i(t) = 0, \tag{15a}$$

$$A_{ij}^*(t) = \frac{1}{N_i^{1/2} N_j^{1/2}} (\mathrm{Bi}^* - \mathrm{Bi}(t)) \cos\mu_i \cos\mu_j, \tag{15b}$$

$$\overline{f}_i = \frac{\Phi_0 \sin\mu_i}{\mu_i N_i^{1/2}}. \tag{15c}$$

From (10a) we then get

$$\overline{T}_{i,l}(t) = \overline{f}_i e^{-\mu_i^2 t} \exp\left[\int_0^t A_{ii}^*(t')\, dt'\right], \tag{16a}$$

where

$$A_{ii}^{*}(t) = \frac{1}{N_i}(\mathrm{Bi}^{*} - \mathrm{Bi}(t))\cos^2\mu_i. \tag{16b}$$

This solution should be used cautiously near the boundary x = 1, since the spectral problem (2) does not correspond completely to the boundary conditions of the problem (1) being solved. To derive improved equations for the flux and temperature at the surface as functions of the average potential, we integrate Eq. (1) in the region V. For the example under consideration, we obtain

$$\left.\frac{\partial\Phi(x, t)}{\partial x}\right|_{x=1} = \frac{d\Phi_{av}(t)}{dt}, \tag{17a}$$

or

$$\Phi(1, t) = -\frac{1}{\mathrm{Bi}(t)}\frac{d\Phi_{av}(t)}{dt}, \tag{17b}$$

where the dimensionless temperature has the form

$$\Phi_{av}(t) = \sum_{i=1}^{\infty}\frac{\bar{f}_i}{\Phi_0}\bar{T}_i(t). \tag{17c}$$

To have a base for comparison, we use the results published in [10], in which Φ_0 = -0.664 and Bi(t) = 1.2 - e^{-t} in the interval from t = 0 to t = 0.4. To illustrate how the end results are affected by the different ways of replacing the variable Biot number by a constant, we consider the cases

1) Bi_0^* = 0.2 (for t = 0);
2) Bi_∞^* = 1.2 (for t = ∞);
3) $\overline{\mathrm{Bi}}^* = \frac{1}{t_f}\int_0^{t_f}\mathrm{Bi}(t)\,dt = a - \frac{b}{t_f}(1 - e^{-t_f})$,

where Bi(t) = a - be^{-t} and t = 0.4.

The results of a numerical calculation of the temperature at the center and on the surface of a plate for selected Biot numbers are given in Table 1. We give both data from [10] and our data from Eqs. (10a) and (12b). The most accurate of these are the results calculated from Eqs. (12b) with allowance for 80 series terms.

One may see that replacing the variable Biot number by a constant Bi = 0.2 provides higher accuracy than in the other two cases. This is because the influence of the nondiagonal matrix elements is more pronounced at the start of the process. These results can be improved by iteration.

It is interesting to note that the solutions obtained from Eq. (10a) are more accurate at the center of the plate. This is explained by the fact that errors originating at the surface die out as they "diffuse" toward the center of the plate. Such a phenomenon has been noted in [10]. In Tables 2 and 3 we give the results for the temperature at the surface and the center of a plate obtained using the DGEAR procedure from the IMSL package [13]. This was done to observe the convergence of the solution from Eq. (12b) as a function of the number of terms. The specified accuracy is reached for the 80 × 80 system, but even the small 20 × 20 system yields fairly accurate results with an insignificant volume of calculations.

Consequently, our approach makes it possible to calculate an important class of problems on the basis of an analytical solution and offers an interesting alternative to the well-known numerical methods.

NOTATION

Bi(t), time dependence of the Biot number in Eq. (13c); Bi^*, characteristic Biot number for the problem (13); d(**x**), coefficient to the linear dissipation term in Eq. (1a); **f(x), potential** distribution at t = 0; K(**x**), coefficient to the diffusion term in Eq. (1a); N_i, normalization integral in the auxiliary problem (2); P(**x**, t), source function in Eq. (1a); t, time or an analytical independent variable; T(**x**, t), distribution of temperature or concentration; $\bar{T}_i(t)$, potential transformed with respect to the integral; w(**x**), coefficient to the nonsteady term (or convection) in Eq. (1a); **x**, position vector; **y**(t), vector of potentials transformed with respect to potentials; $\alpha(\mathbf{x}, t)$, time dependence of the coefficients of the boundary condition, as in Eq. (1b); $\alpha^*(\mathbf{x})$, characteristic coefficient of the boundary condition; $\beta(\mathbf{x})$, coefficient of the boundary condition, as in Eq. (1b); $\Phi(\mathbf{x}, t)$, inhomogeneous term in the boundary condition, Eq. (1b); $\psi(\mu_i, \mathbf{x})$, eigenfunction in the auxiliary problem (2); $k_i(\mathbf{x})$, symmetric kernel in the integral transformation pair (3a), (3b); μ_i, eigenvalues in the auxiliary problem (2); $\Phi(\mathbf{x}, t)$, distribution of dimensionless temperature in the problem (13); Φ_0, dimensionless initial temperature in the problem (13). Indices: overbar, quantity transformed relative to the integral; i, j, orders of the eigenvalues, i, j = 1, 2, ...; ℓ, low-order solution; h, low-order iterative solution.

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